J Eng Math (2007) 59:385–398
DOI 10.1007/s10665-007-9171-5

Acoustic scattering by a circular semi-transparent conical surface

Mikhail Anatolievich Lyalinov · Ning Yan Zhu

Received: 21 June 2007 / Accepted: 5 July 2007 / Published online: 16 August 2007

Abstract The scattering of a plane acoustic wave by a circular semi-transparent conical surface with impedance-type boundary conditions is studied. The analytic solution is constructed on the basis of the incomplete separation of variables and the reduction of the problem to a functional difference equation of the second order. Although the latter is equivalent to a Carleman boundary-value problem for analytic vectors, the solution is studied by means of the direct reduction method, that is, converting the functional difference equations to a Fredholm-type integral equation. Its unique solvability is then studied and the expression for the scattering amplitude of the spherical wave from the vertex is discussed. Some numerical results for axial incidence are also presented.

Keywords Carleman problems · Fredholm integral equation · Functional difference equation · Scattering amplitude · Wiener–Hopf

1 Introduction

The theory of diffraction by a cone with ideal boundary conditions has been developed in numerous papers [1–6] and others. The essential progress in the ideal boundary conditions is now obvious. In particular, in the works of Smyshlyaev, Babich and their co-authors numerical results for cones of arbitrary shape have been obtained.

The impedance-type boundary conditions for cone problems were considered in [7–11]. The corresponding class of problems is more complex, in particular, because it leads to a study of functional difference equations. The theory of that kind of equations is closely related to some boundary-value problems for analytic vectors.[1] It is well known that application of the Fourier transformation to diffraction problems with semi-infinite "rectangular" geometries (like diffraction by a half-plane) leads, by means of the Wiener–Hopf reduction procedure, to a Riemann–

[1] In Appendix B we demonstrate the direct reduction of the second-order functional difference equations to a Carleman problem for analytic vectors.

M. A. Lyalinov (✉)
Department of Mathematical Physics, Institute of Physics, Saint-Petersburg University, Ulianovskaya 1-1,
Petershof 198504, Russia
e-mail: lyalinov@yandex.ru

N. Y. Zhu
Institut für Hochfrequenztechnik, Universität Stuttgart, Pfaffenwaldring 47, 70550 Stuttgart, Germany
e-mail: zhu@ihf.uni-stuttgart.de

Hilbert boundary-value problem for analytic functions (vectors). However, recently it has been demonstrated [12] that the Laplace transformation applied to the diffraction problem in a wedge-shaped domain (like the Malyuzhinets problem) enables one to obtain generalized Wiener–Hopf equations which are then reducible to standard Wiener–Hopf equations. Actually Daniele has demonstrated the equivalence of the problem for the Malyuzhinets functional difference equations to that for the Wiener–Hopf equations. Some functional equations were also exploited in [13] for splitting some Wiener–Hopf kernels. It is worth noting that in [14] the problem for the second-order functional difference equation is reduced to a Riemann–Hilbert problem on a Riemann surface. Application of the Kontorovich–Lebedev transformation and of the incomplete separation of variables to the diffraction by a cone with impedance-type boundary conditions [7,8] enables one to obtain the second-order functional difference equation which is a very natural tool also for the problem at hand. In [15,16] the system of Malyuzhinets' functional equations, obtained in the framework of the Sommerfeld–Malyuzhinets integral transformation, reduces to a second-order functional difference equation and then directly to the integral equations (see also [17]). It seems that this approach is an alternative to that in [12] and has some technical advantages because, contrary to the Wiener–Hopf approach, it exploits a class of meromorphic functions (i.e., those having no branch points). In the present paper we study diffraction of a plane acoustic wave by a semi-transparent conical surface; we apply and develop the approaches of the papers [7] and [10]. The paper is organized as follows. We formulate the scattering problem and partially separate the spherical variables. We obtain the functional difference equations for the Fourier coefficients of the corresponding series. The functional difference equations are then transformed to a set of Fredholm integral equations. Their properties and some other mathematical aspects of the problem at hand are studied in Appendices. We then give a formula for the diffraction coefficient in the domain not illuminated by the rays reflected from the conical surface. The numerical results are presented for different cone openings and surface impedances.

2 Formulation

Let a circular conical surface C divide the 3D space into two parts: the "exterior" part Ω_1 $(\theta < \theta_1)$ and the "interior" part Ω_2 $(\theta > \theta_1)$, see Fig. 1, where the spherical coordinates (r, θ, φ) are related to the Cartesian ones (x_1, x_2, x_3) via

$$x_1 = r \sin\theta \cos\varphi, \quad x_2 = r \sin\theta \sin\varphi, \quad x_3 = r \cos\theta, \tag{1}$$

where $0 \le \varphi < 2\pi,\ 0 \le \theta \le \pi,\ r \ge 0$. The axis Ox_3 coincides with the axis of symmetry (not shown in Fig. 1), $\theta = \theta_1$ is the equation of the conical surface with $\pi/2 < \theta_1 < \pi$.

The incident plane wave field in spherical coordinates is given by

$$U^i = \mathrm{e}^{-\mathrm{i}kr\cos\hat{\theta}(\omega,\omega_0)}, \quad k = \Omega/c, \tag{2}$$

where $\omega_0 = (\theta_0, \varphi_0)$ is the unit vector attached to the direction of incidence, $\omega = \vec{r}/r = (\theta, \varphi)$, $\cos\hat{\theta}(\omega, \omega_0) = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0)$, c is the wave speed in the acoustic medium. The harmonic dependence on time $\mathrm{e}^{-\mathrm{i}\Omega t}$ is omitted throughout the paper. We also assume that the incident wave illuminates completely the conical surface from Ω_1, that is, $\theta_0 < \pi - \theta_1$.

We look for the classical solution of the problem. The wave field $\hat{U}(kr, \omega, \omega_0) = U(kr, \omega, \omega_0) + U^i(kr, \omega, \omega_0)$ in Ω_1 and $V(kr, \omega, \omega_0)$ in Ω_2 satisfy the Helmholtz equation

$$\left(\triangle + k^2\right)\{U, V\} = 0, \tag{3}$$

and the boundary conditions

$$\frac{1}{r}\frac{\partial\left(U + U^i\right)}{\partial\theta}\bigg|_C - \mathrm{i}k\eta\left[\left(U + U^i\right) - V\right]\bigg|_C = 0, \tag{4}$$

$$\frac{\partial\left(U + U^i\right)}{\partial\theta}\bigg|_C = \frac{\partial V}{\partial\theta}\bigg|_C, \tag{5}$$

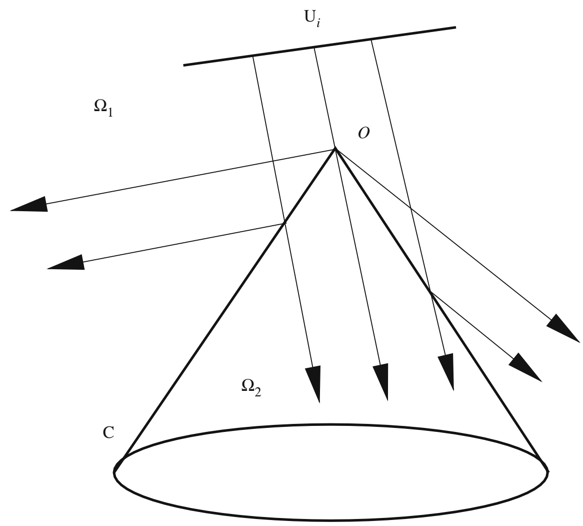

Fig. 1 Diffraction by a semi-transparent cone

where η is the impedance of the surface which is independent of k and is bounded. We assume that $\mathfrak{Re}\,\eta > 0$, i.e., the conical surface is absorbing. Equation 5 shows that the displacement $\xi = \frac{1}{\rho\Omega^2}\frac{\partial\hat{U}}{\partial\theta}$ is continuous across the conical surface, whereas condition (4) specifies the displacement due to the difference of the acoustic pressure outside and inside the conical surface; ρ is the density of the surrounding medium. This type of conditions represents a simple model of a thin semi-transparent surface. These conditions are traditionally deduced by means of an asymptotic procedure applied to a canonical boundary-value problem for a thin curvilinear layer separating acoustic media (see, e.g., [18,19]). The leading terms of the corresponding expansions enable one to conclude that the boundary conditions (4) and (5) are valid in the leading approximation. It is easily demonstrated that, owing to the boundary conditions, the rays of the incident wave penetrate through the semi-transparent conical surface without change of the propagation direction and are reflected by the surface in accordance with geometrical optics; see Fig. 1.

Meixner's condition is satisfied ($r \to 0$)

$$|U| \le \text{const}\, r^h, \quad r\,|\nabla U| \le \text{const}\, r^h, \; h > -1/2, \tag{6}$$

uniformly with respect to the angular variables.[2] Similar conditions are valid for V.

Consider a unit sphere centred at the vertex O of the cone. The conical surface C separates the sphere into two parts M_1 and M_2 with the boundary σ. We introduce the length of the geodesics (broken for the reflected wave, see also [6,11] for definitions of $\hat{\theta}'(\omega,\omega_0)$) on the unit sphere

$$\hat{\theta}'(\omega,\omega_0) = \inf_{\omega'\in\sigma}(\hat{\theta}(\omega',\omega_0) + \hat{\theta}(\omega,\omega')), \tag{7}$$

where $\hat{\theta}(\omega',\omega_0)$ is the distance between the points ω_0 and ω' on the unit sphere. The domain M_1 can be also subdivided into two parts $M_1' = \{\theta,\varphi : \; \hat{\theta}'(\omega,\omega_0) > \pi\}$, the angular domain, where only the spherical wave from the vertex propagates at infinity and $M_1'' = \{\theta,\varphi : \; \hat{\theta}'(\omega,\omega_0) < \pi\}$, the subdomain being illuminated, in addition, by the rays reflected from the cone. The common boundary l of M_1' and M_1'' is the curve of the so-called singular directions. The details of the description above and the corresponding diagram can be found in [6]. In the domain Ω_1 for those directions satisfying

$$\hat{\theta}'(\omega,\omega_0) > \pi, \tag{8}$$

(the so-called "oasis") the wave field at infinity has the form

[2] The constant h solves a transcendental equation deduced in a traditional manner and, in particular, depends on the cone opening angle. The explicit value of h is not exploited in the analysis below.

$$U^s(r,\omega,\omega_0) = D_u(\omega,\omega_0)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{-\mathrm{i}kr}\,(1+O(1/kr)),\quad kr \to \infty, \tag{9}$$

where $D_u(\omega,\omega_0)$ is a smooth function of ω provided the inequality (8) is valid. In the sub-domain

$$\hat\theta'(\omega|_C,\omega_0) < \hat\theta'(\omega,\omega_0) < \pi \tag{10}$$

the waves reflected from the cone and surface[3] waves should be present. The singular directions form a surface in Ω_1

$$\hat\theta'(\omega,\omega_0) = \pi. \tag{11}$$

Remark 1 In the same manner, in the domain Ω_2 for those directions satisfying

$$\pi > \hat\theta(\omega,\omega_0) \ge \hat\theta(\omega|_C,\omega_0), \tag{12}$$

from the scattered field of pressure V, beside the transmitted wave, we can extract the part related to the spherical wave scattered by the vertex

$$V^s(r,\omega,\omega_0) = D_v(\omega,\omega_0)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{-\mathrm{i}kr}\,(1+O(1/kr)), \tag{13}$$

where $D_v(\omega,\omega_0)$ is a smooth function.

The total field in Ω_2 is the sum of the transmitted and spherical diffracted waves. The contribution of the surface waves is negligible outside a close neighbourhood ($\hat\theta(\omega,\omega_0) \sim \hat\theta(\omega|_C,\omega_0)$) of the conical surface. The vicinity of the direction $\hat\theta(\omega,\omega_0) \sim \pi$ is the zone of the complex interference behaviour in Ω_2.

3 Kontorovich–Lebedev transform and problem for the spectral functions

The solution of the problem is based on the incomplete separation of variables. First, we separate the radial variable and look for the solution in the form of Kontorovich–Lebedev integrals. For the incident wave we have [8]

$$U^i(kr,\omega,\omega_0) = \frac{4}{\mathrm{i}\sqrt{2\pi}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \nu\sin(\pi\nu)\,u^i_\nu(\omega,\omega_0)\,\frac{K_\nu(-\mathrm{i}kr)}{\sqrt{-\mathrm{i}kr}}\,\mathrm{d}\nu, \tag{14}$$

where

$$u^i_\nu(\omega,\omega_0) = -\frac{P_{\nu-1/2}\left(-\cos\hat\theta(\omega,\omega_0)\right)}{4\cos(\pi\nu)}. \tag{15}$$

Here $P_{\nu-1/2}(x)$ is the Legendre function, $K_\nu(z)$ is the modified Bessel (Macdonald) function

$$K_\nu(z) = -\mathrm{i}\,\frac{\pi}{2}\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}\,H^{(2)}_\nu(-\mathrm{i}z).$$

The integral (14) converges exponentially provided

$$\hat\theta(\omega,\omega_0) > \frac{\pi}{2} + |\arg(-\mathrm{i}k)|\,,\quad |\arg(-\mathrm{i}k)| \le \pi/2. \tag{16}$$

The latter is easily verified by means of the asymptotics (e.g. [20])

$$K_\nu(z) \sim \mathrm{const}\,\frac{\nu^\alpha\cos\left[\nu\left(\pi/2+|\arg(z)|\right)\right]}{\sin(\pi\nu)},$$

as $|\mathfrak{Im}\,\nu| \to \infty$, $|\arg(\nu)| \to \pi/2$, $\alpha = |\mathfrak{Re}\,\nu| - 1/2$, and ([21])

$$P^{-|n|}_{\nu-1/2}(\cos\theta) = \frac{\sqrt{2}\,\Gamma(\nu-|n|+1/2)}{\sqrt{\pi\sin\theta}\,\Gamma(\nu+1)}\,\cos\left(\nu\theta-|n|\,\pi/2-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{\nu}\right)\right),$$

[3] The surface wave can be excited if $\mathfrak{Im}\,\eta < 0$.

where $|n|$ is bounded, $\delta < \theta < \pi - \delta$, $\delta > 0$ is small, $|\nu| \gg 1/\delta$, $|\arg(\nu)| < \pi$.

In view of the representation (14), it is reasonable to seek a solution in the form [8]

$$U(kr, \omega, \omega_0) = \frac{4}{\mathrm{i}\sqrt{2\pi}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \nu \sin(\pi\nu)\, u_\nu(\omega, \omega_0) \frac{K_\nu(-\mathrm{i}\,kr)}{\sqrt{-\mathrm{i}\,kr}}\, \mathrm{d}\nu, \tag{17}$$

where the integral converges provided

$$\hat{\theta}'(\omega, \omega_0) > \pi/2 + |\arg(-\mathrm{i}\,k)|, \quad |\arg(-\mathrm{i}\,k)| \le \pi/2, \tag{18}$$

$$V(kr, \omega, \omega_0) = \frac{4}{\mathrm{i}\sqrt{2\pi}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \nu \sin(\pi\nu)\, v_\nu(\omega, \omega_0) \frac{K_\nu(-\mathrm{i}\,kr)}{\sqrt{-\mathrm{i}\,kr}}\, \mathrm{d}\nu \tag{19}$$

with the conditions (16).

Now we formulate the conditions which specify the class of spectral functions u_ν, v_ν in (17), (19) and ensure that the representations solve the problem at hand. For compactness we give them for u_ν; those for v_ν are similar.

Assume that $u_\nu(\omega, \omega_0)$ is twice continuously differentiable on $\omega = (\theta, \varphi)$, and has a continuous normal derivative on σ for any ν, in the strip $\Pi_\delta = \{\nu : |\mathfrak{Re}\,\nu| < 1 + \delta\}$, $\delta > 0$, and is small. Besides, u_ν is even and is regular in the strip Π_δ as well as the derivative $\frac{\partial u_\nu}{\partial\theta}|_\sigma$. Let the spectral functions solve the problem for the equations

$$\left(\triangle_\omega + (\nu^2 - 1/4)\right) u_\nu(\omega, \omega_0) = 0, \tag{20}$$

$$\left(\triangle_\omega + (\nu^2 - 1/4)\right) v_\nu(\omega, \omega_0) = 0 \tag{21}$$

with the boundary conditions

$$\left.\frac{\partial \hat{U}_{\nu+1}}{\partial\theta}\right|_\sigma - \left.\frac{\partial \hat{U}_{\nu-1}}{\partial\theta}\right|_\sigma = (-2\nu)\eta \left.\left(\hat{U}_\nu - v_\nu\right)\right|_\sigma, \tag{22}$$

$$\left.\frac{\partial \hat{U}_{\nu+1}}{\partial\theta}\right|_\sigma - \left.\frac{\partial \hat{U}_{\nu-1}}{\partial\theta}\right|_\sigma = \left.\frac{\partial v_{\nu+1}}{\partial\theta}\right|_\sigma - \left.\frac{\partial v_{\nu-1}}{\partial\theta}\right|_\sigma, \tag{23}$$

where $\hat{U}_\nu = u_\nu + u_\nu^i$, $\triangle_\omega$ is the Laplace–Beltrami operator on S^2, $\triangle_\omega = \dfrac{1}{\sin\theta}\,\partial_\theta\,(\sin\theta\,\partial_\theta) + \dfrac{1}{\sin^2\theta}\,\partial_\varphi^2$.

The integral (17) converges provided $u_\nu(\omega, \omega_0)$ satisfies the estimate in the strip Π_δ

$$|u_\nu(\omega, \omega_0)| \le \text{Const}\, \frac{|\nu|^\kappa}{|\cos[(\pi + \epsilon)\nu]|}, \quad \hat{\theta}'(\omega, \omega_0) > \pi, \;\; \kappa \ge 0, \tag{24}$$

$\epsilon > 0$ is small. Provided $\hat{\theta}'(\omega, \omega_0) \le \pi$, we assume that

$$|u_\nu(\omega, \omega_0)| \le \text{Const}\, \frac{|\nu|^{\kappa_1}}{|\cos[\nu\hat{\theta}'(\omega, \omega_0)]|}, \quad \kappa_1 \ge -1/2. \tag{25}$$

Analogous conditions are met for v_ν.

The reason for considering the problem (20–23) in the specified class of functions is as follows. Provided the spectral functions satisfy the problem (20–23), the integral representations converge and can be substituted in the Helmholtz equations and the boundary conditions. Moreover, the Meixner conditions and conditions at infinity are also valid. Indeed, by the direct substitution of (14), (17), (19) in (3), with the aid of the Bessel equation for $K_\nu(z)$ and (20), (21) we verify (see also [22]) that $U(kr, \omega, \omega_0)$, $V(kr, \omega, \omega_0)$ satisfy the Helmholtz equation.

Let us turn to the boundary condition (4) and substitute the representation (14), (17), (19). On taking into account that

$$\frac{K_\nu(z)}{z} = \frac{K_{\nu+1}(z) - K_{\nu-1}(z)}{2\nu},$$

we have

$$\int_{-i\infty}^{i\infty} \frac{\nu \sin(\pi\nu)}{\sqrt{-ikr}} \left[\left.\frac{\partial \hat{U}_\nu}{\partial \theta}\right|_\sigma (-ik)\, \frac{K_{\nu+1}(-ikr) - K_{\nu-1}(-ikr)}{2\nu} - ik\eta(\hat{U}_\nu - v_\nu)\Big|_\sigma K_\nu(-ikr) \right] d\nu = 0.$$

Then

$$\frac{-ik}{\sqrt{-ikr}} \left\{ \int_{-i\infty-1}^{i\infty-1} \frac{\sin(\pi\nu)}{2} \left.\frac{\partial \hat{U}_{\nu+1}}{\partial\theta}\right|_\sigma K_\nu(-ikr)\, d\nu - \int_{-i\infty+1}^{i\infty+1} \frac{\sin(\pi\nu)}{2} \left.\frac{\partial \hat{U}_{\nu-1}}{\partial\theta}\right|_\sigma K_\nu(-ikr)\, d\nu \right.$$
$$\left. + \int_{-i\infty}^{i\infty} \frac{\sin(\pi\nu)}{2}\, 2\eta\, \nu\, (\hat{U}_\nu - v_\nu)\Big|_\sigma K_\nu(-ikr)\, d\nu \right\} = 0$$

and finally

$$\frac{-ik}{\sqrt{-ikr}} \left(\int_{-i\infty}^{i\infty} \frac{\sin(\pi\nu)}{2} \left[\left.\frac{\partial \hat{U}_{\nu+1}}{\partial\theta}\right|_\sigma - \left.\frac{\partial \hat{U}_{\nu-1}}{\partial\theta}\right|_\sigma + 2\eta\, \nu\, (\hat{U}_\nu - v_\nu)\Big|_\sigma \right] K_\nu(-ikr)\, d\nu = 0,$$

which implies the validity of (22). When deforming the contours into the imaginary axis, we exploited the regularity of $\frac{\partial \hat{U}_\nu}{\partial\theta}|_\sigma$ in the strip Π_δ. The boundary condition (5) is similarly treated. The boundary-value problem (20–23) is non-traditional because it is non-local with respect to the spectral variable in view of the boundary conditions connecting $u_\nu|_\sigma$, $v_\nu|_\sigma$ and their normal derivatives with a shift in ν.

Let the spectral functions $u_\nu(\omega, \omega_0)$, $v_\nu(\omega, \omega_0)$ solve the problem (20–23); then the Kontorovich–Lebedev integral representations fulfil the Meixner conditions and, in particular, as $kr \to \infty$, the acoustic pressure U in the "oasis", (i.e., in the domain not illuminated by the reflected rays, $\hat{\theta}'(\omega, \omega_0) > \pi$) has the asymptotic form

$$U(kr, \omega, \omega_0) = D(\omega, \omega_0)\, \frac{e^{ikr}}{-ikr} \left(1 + O\left(\frac{1}{kr}\right)\right), \tag{26}$$

where

$$D(\omega, \omega_0) = \frac{2}{i} \int_{-i\infty}^{i\infty} \nu \sin(\pi\nu)\, u_\nu(\omega, \omega_0)\, d\nu \tag{27}$$

is the scattering amplitude (diagram). The analogue of this formula has been obtained in [4] for an ideal cone.

This characteristic function for the diffraction of a spherical wave from the vertex is of most importance in practice. It is obvious that the diagram depends globally on the spectral function, i.e., on the whole solution of the problem on the spectral function.

3.1 Incomplete separation of the spherical variables and functional difference equations

We further reduce the boundary-value problem for the spectral function by taking into account the symmetry of the boundary σ. We separate the angular variables thus

$$u_\nu(\omega, \omega_0) = \sum_{n=-\infty}^{+\infty} i^n\, e^{-in\varphi}\, R_u(\nu, n)\, \frac{P_{\nu-1/2}^{-|n|}(\cos\theta)}{d_{\theta_1} P_{\nu-1/2}^{-|n|}(\cos\theta_1)}, \tag{28}$$

$$v_\nu(\omega, \omega_0) = \sum_{n=-\infty}^{+\infty} i^n\, e^{-in\varphi}\, R_v(\nu, n)\, \frac{P_{\nu-1/2}^{-|n|}(-\cos\theta)}{d_{\theta_1} P_{\nu-1/2}^{-|n|}(-\cos\theta_1)}, \tag{29}$$

where $\theta = \theta_1$ is the equation of σ on the sphere S^2, whereas for u_ν^i we have

$$u_\nu^i(\omega, \omega_0) = \sum_{n=-\infty}^{+\infty} i^n\, e^{-in\varphi}\, R_i(\nu, n)\, \frac{P_{\nu-1/2}^{-|n|}(-\cos\theta)}{d_{\theta_1} P_{\nu-1/2}^{-|n|}(-\cos\theta_1)}, \quad 0 < \theta_0 < \theta, \;\; \varphi_0 = 0, \tag{30}$$

where

$$R_i(\nu,n) \;=\; \frac{\mathrm{i}^n}{-4\cos(\pi\nu)}\,\frac{\Gamma(\nu+|n|+1/2)}{\Gamma(\nu-|n|+1/2)}\,d_{\theta_1} P^{-|n|}_{\nu-1/2}(-\cos\theta_1)\, P^{-|n|}_{\nu-1/2}(\cos\theta_0)$$

with $d_\theta := \frac{\mathrm{d}}{\mathrm{d}\theta}$.

The series (28), (29) solve Eqs. 20, 21; then substituting them in the boundary conditions (22), (23), we have

$$R_u(\nu+1,n) - R_u(\nu-1,n) = (-2\mathrm{i})\eta\,[w(\nu,n)R_u(\nu,n) - w_t(\nu,n)R_v(\nu,n)] + S_i(\nu,n) \tag{31}$$

and

$$R_u(\nu+1,n) - R_u(\nu-1,n) = R_v(\nu+1,n) - R_v(\nu-1,n) + T_i(\nu,n), \tag{32}$$

where

$$S_i(\nu,n) = -[R_i(\nu+1,n) - R_i(\nu-1,n)] - 2\mathrm{i}\,\eta\, w_t(\nu,n)R_i(\nu,n),$$

$$T_i(\nu,n) = -R_i(\nu+1,n) + R_i(\nu-1,n)$$

are known functions specified by the incident wave,

$$w(\nu,n) \;=\; -\mathrm{i}\nu\,\frac{P^{-|n|}_{\nu-1/2}(\cos\theta_1)}{d_{\theta_1}P^{-|n|}_{\nu-1/2}(\cos\theta_1)},\quad w_t(\nu,n) \;=\; -\mathrm{i}\nu\,\frac{P^{-|n|}_{\nu-1/2}(-\cos\theta_1)}{d_{\theta_1}P^{-|n|}_{\nu-1/2}(-\cos\theta_1)}.$$

We remark that

$$w(\nu,n) > 0,\ \text{and}\ w(\nu,n) \;=\; 1 \,+\, O_n\left(\frac{1}{\nu}\right),\quad \nu\to\mathrm{i}\infty, \tag{33}$$

where $|n|$ is fixed. The same is valid for $-w_t$.

The problem at hand will be solved if one determines the unknown functions $R_{u,v}(\nu,n)$ from the system of functional difference equations (31), (32). The solutions are sought among even meromorphic functions which are regular in the strip Π_δ and satisfy the estimates

$$|R_u(\nu,n)| \;\le\; \text{const}\,|\nu|^{1/2}|\exp[\mathrm{i}\nu(\theta_1-\theta_0)]|, \tag{34}$$

$$|R_v(\nu,n)| \;\le\; \text{const}\,|\nu|^{1/2}|\exp[\mathrm{i}\nu(\theta_1-\theta_0)]| \tag{35}$$

in this strip, $\nu\to\mathrm{i}\infty$.

It seems that the system of equations (31), (32) cannot be solved explicitly, although the second equation is rather simple and is equivalent to

$$R_u(\nu,n) + R_i(\nu,n) = R_v(\nu,n) \tag{36}$$

in the corresponding class of functions, which can be demonstrated by use of the Fourier transformation along imaginary axis.

Taking into account Eq. 36, we reduce (31) to the form

$$R_u(\nu+1,n) - R_u(\nu-1,n) = -2\mathrm{i}\,\eta W(\nu,n)R_u(\nu,n) + S(\nu,n) \tag{37}$$

with

$$S(\nu,n) = -\,[R_i(\nu+1,n) - R_i(\nu-1,n)]\,,\quad W(\nu,n) = w(\nu,n) - w_t(\nu,n).$$

Equations very similar to (37) were studied in [7,10]; see also [15]. Taking into account the important findings of [7], we follow mainly the methodology of [10]. Equation 37, a functional difference equation of the second order, is the main subject of our further studies. Having solved this equation (for example, numerically after reduction to an integral equation), one can compute the solution of the problem and all important characteristics of the far field such as scattering amplitudes.

4 Direct reduction to a Fredholm integral equation

By use of a known procedure (e.g., [23]; see also [19, Chapter 6]) we invert the difference operator at the left-hand side of (37) and, by taking into account that $R_u(\nu, n) = R_u(-\nu, n)$, arrive at the integral representation

$$R_u(\nu, n) \;=\; \eta \int_0^{\mathrm{i}\infty} \frac{W(t,n)\,R_u(t,n)\,\sin(\pi t)}{\cos(\pi t)+\cos(\pi\nu)}\,\mathrm{d}t \;-\; R_i(\nu, n). \tag{38}$$

The representation (38) enables one to determine $R_u(\nu, n)$ in the strip $|\Re\mathfrak{e}\,\nu| < 1$ provided the value of $R_u(t, n)$ is known on the positive imaginary axis. Letting $\nu \in \mathrm{i}\mathbb{R}$ (i.e., on the imaginary axis) at the left-hand side of (38), we arrive at an integral equation for $R_u(t, n)$. It can be demonstrated that the integral equation is, in some appropriate sense, *equivalent* to the functional difference equation in the corresponding class of functions.

A remarkable property of the integral equation (38) is that its solution is *unique* ($\Re\mathfrak{e}\,\eta > 0$) in the respective class of functions [7,10]. In Appendix A we demonstrate subsequently that the integral equation possesses the Fredholm property (its operator can be represented as a sum of boundedly invertible and compact operators); then its solvability follows from uniqueness, which is a standard trick for this kind of equation.

The Fredholm property (Appendix A) is of principal importance for the integral equation at hand, because, apart from its unique solvability, it ensures convergence and stability of the reasonable numerical procedures for the solution.

5 Alternative integral representations of the solution for positive *k*

The Kontorovich–Lebedev representations of the solution U, V converge in conditions (16), (18), i.e., the wave number k is complex. However, one can get an "analytic continuation" for real k. It is especially important for studying the far-field asymptotics [10]. Although we do not exhaustively consider the formulae for the far field herein, an alternative representation for the solution for real k can be found on the basis of the regularized form of the Kontorovich–Lebedev integrals [24]. However, other approaches are also possible.

The integral representation for the wave field with the Macdonald function admits modification to such a form which is valid for real k without limitations on $\hat{\theta}'(\omega, \omega_0)$[4]

$$U(kr, \omega, \omega_0) \;= 4\mathrm{i}\sqrt{\frac{\pi}{2}} \int_{\infty \mathrm{e}^{-\mathrm{i}\Phi}}^{\infty \mathrm{e}^{\mathrm{i}\Phi}} \nu \mathrm{e}^{-\mathrm{i}\pi\nu/2} u_\nu(\omega, \omega_0) \frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}} \mathrm{d}\nu, \tag{39}$$

where $\Phi \in (0, \pi/2)$, whereas the contour of integration comprises those singularities of $u_\nu(\omega, \omega_0)$ on ν which are located on the real positive axis. The integral representation (39) is an analytic continuation of the representation (17) on real values of k and any $\omega \in S^2\backslash\Sigma$. Indeed, in conditions (18) we exploit the correlation

$$\sin\pi\nu K_\nu(-\mathrm{i}kr) = \frac{\pi}{2}[\mathrm{e}^{\mathrm{i}\pi\nu/2} J_{-\nu}(kr) - \mathrm{e}^{-\mathrm{i}\pi\nu/2} J_\nu(kr)]$$

and transform the integral (17) to the equivalent form

$$U(kr, \omega, \omega_0) \;= 4\mathrm{i}\sqrt{\frac{\pi}{2}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \nu \mathrm{e}^{-\mathrm{i}\pi\nu/2} u_\nu(\omega, \omega_0) \frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}} \mathrm{d}\nu, \tag{40}$$

where the integral converges exponentially. Provided the modulus of $\nu = |\nu|\mathrm{e}^{\mathrm{i}\phi}$, $(|\phi| < \pi/2)$ is sufficiently large, $J_\nu(kr) \sim (kr/2)^\nu/\Gamma(\nu+1)$, and the estimate

$$\begin{aligned}|\mathrm{e}^{-\mathrm{i}\pi\nu/2} u_\nu(\omega, \omega_0) J_\nu(kr)| \;\le\;& C\,\exp[-|\nu|\log|\nu|\cos\phi \\ & -|\nu|(\sin\phi(\arg k - \pi/2 - \phi) + |\sin\phi|\hat{\theta}'(\omega, \omega_0) - \cos\phi[1+\log(|k|r/2)])]\end{aligned}$$

[4] This observation is due to V.P. Smyshlyaev, Bath University, UK.

is valid, where $|kr|$ is arbitrarily fixed. For sufficiently large $|\nu|$, $\log|\nu| > 1 + \log(|k|r/2)$ we deform the contour in (40) into $C_\phi = (e^{-i\phi}\infty, e^{i\phi}\infty)$, where $\phi \in [0, \pi/2)$ and get the exponentially convergent integral (39) for real k and any $\omega \in S^2$.

Another way is to use the Sommerfeld representation (see also [8])

$$U(r,\omega,\omega_0) = \frac{1}{\sqrt{-ikr}}\frac{1}{2i\pi}\int_\gamma e^{-ikr\cos\alpha}\Phi(\alpha,\omega,\omega_0)d\alpha, \tag{41}$$

where γ is the double-loop Sommerfeld contour,

$$\Phi(\alpha,\omega,\omega_0) = \sqrt{2\pi}\int_{-i\infty}^{+i\infty} \nu\, u_\nu(\omega,\omega_0)(e^{i\nu\alpha} - e^{-i\nu\alpha})/2d\nu. \tag{42}$$

Similar formulae are valid for the pressure V. The formulae (41), (42) are well suited for the derivation of the far field. The contour of integration in (41) is deformed into the steepest-descent paths and the singularities of the integrand in the complex plane α can be captured. The branch points are responsible for the reflected wave, the poles are for the surface waves. The saddle points $\pm\pi$ give rise to the spherical wave from the vertex (see [10]). A detailed study of the far field will be discussed elsewhere. For example, in the axi-symmetric case ($\theta_0 = 0$), for the scattering amplitude in the "oasis" ($0 \le \theta < 2\theta_1 - \pi$) one has

$$D(\omega,\omega_0) = \frac{2}{i}\int_{-i\infty}^{i\infty} \nu\sin(\pi\nu)\; R_u(\nu,0)\,\frac{P_{\nu-1/2}(\cos\theta)}{d_{\theta_1}P_{\nu-1/2}(\cos\theta_1)}\,d\nu, \tag{43}$$

where $R_u(\nu,0)$ is determined from the Fredholm integral equation (38) and the integral in (43) converges rapidly. The value $\theta_{\text{sing}} = 2\theta_1 - \pi$ corresponds to the singular direction in which the scattering amplitude has a singularity and formula (43) is not applicable. In these directions the far wave field is described by parabolic cylinder functions.

6 Numerical aspects and examples

To solve the Fredholm integral equation (38), the asymptotic behaviour of $R_u(\nu,0)$ has been taken into account explicitly via $R_u(\nu,0) = \rho_u(\nu)\exp(i\nu\theta_1)$. Then the semi-infinite domain of integration has been transformed into a finite one with the aid of $\nu = \dfrac{ip}{\theta_1}\log\dfrac{1+\xi}{1-\xi}$, $p \gg 1$. A quadrature method based on Gauss–Legendre scheme is applied to the resulting integral equation for $\rho_u(\nu)$, leading to its fast and accurate solution.

The scattering diagram $D(\omega,\omega_0)$ is obtained by evaluating the integral expression (43) using a 20-point Gauss–Laguerre scheme. The values of $\rho_u(\nu)$ have been calculated via an integral extrapolation.

Figure 2 displays the amplitude of the scattering diagram as a function of the angle θ and the impedance η. In the Neumann case with $\eta = 0$, the solution is exact and agrees with that given in [9]. The larger the absolute value of η, the more transparent does the cone become, and hence the weaker the scattering diagram will be. As θ approaches the singular direction θ_{sing}, the scattering diagram increases.

The dependence of the scattering diagram upon the opening angle of the cone for a certain impedance η is shown in Fig. 3. With decreasing opening angle θ_1 the range of the oasis becomes smaller and is also closer to the singular direction. As $\theta_1 \to \pi/2$, the far-field asymptotics (26), and hence the scattering diagrams lose their validity.

The results shown in Figs. 2 and 3 have been obtained using 100 abscissae in the Gauss–Legendre scheme, $p = 20$.

7 Conclusion

The proposed approach based on incomplete separation of variables enabled us to reduce the problem to functional difference equations and then to Fredholm integral equations. The numerical solution of the integral equations has been used for evaluation of the scattering diagram for the wave scattered from the cone's vertex. To that end

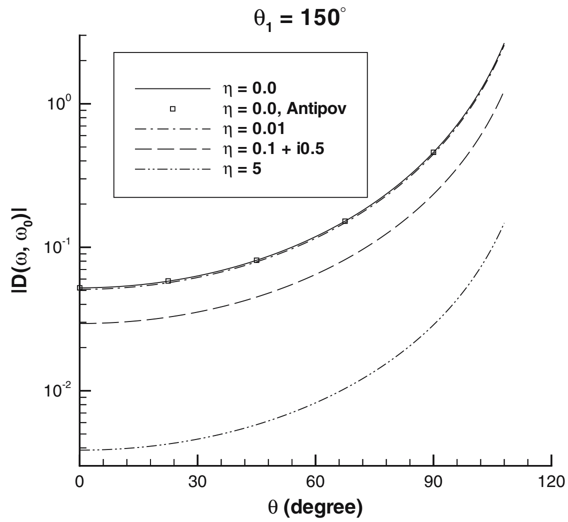

Fig. 2 Scattering diagram for different impedances

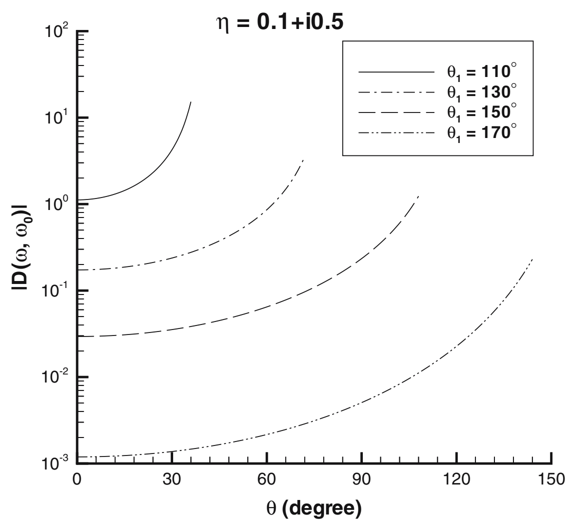

Fig. 3 Scattering diagram for different cone-openings

we exploited the explicit integral representation of the scattering diagram in the "oasis", i.e., in the domain not illuminated by reflected or transmitted rays.

By means of alternative integral representations of the solution one can also deduce the expressions of waves reflected and transmitted by the conical surface, the surface waves which may arise for $\Im\mathfrak{m}\,\eta < 0$ [10] and the spherical wave outside the "oasis". However, to this end some additional technical tools should be developed. The numerical results in the "oasis" for the non-axial incidence, as well as those for the corresponding electromagnetic analogue of the problem at hand, have been also obtained. We expect to consider these questions and to report the mentioned results in forthcoming publications.

Acknowledgements The authors are thankful to the anonymous reviewers for their helpful suggestions. One of the authors (MAL) was supported in part by the NATO Science-for-Peace Grant—CBP.MD.SFPP 982376. The authors are grateful to the guest-editors I.D. Abrahams and J.B. Lawrie for their kind invitation to submit a paper to the special issue.

Appendix A: Fredholm property

It is convenient to reduce the integral equation (38) by introducing the new variable $x = 1/\cos(\pi\nu)$ and a new unknown function

$$\mathbf{r}(x,n) = R_u(\nu,n)\,\cos(\pi\nu)|_{x=1/\cos(\pi\nu)}\,.$$

We have

$$\mathbf{r}(x,n) + \frac{\eta}{\pi}\int_0^1 \frac{\mathcal{W}(y,n)\,\mathbf{r}(y,n)\mathrm{d}y}{x+y} = \mathbf{r}_i(x,n), \tag{44}$$

where

$$\mathbf{r}_i(x,n) = -R_i(\nu,n)\,\cos(\pi\nu)|_{x=1/\cos(\pi\nu)},$$
$$\mathcal{W}(x,n) = W(\nu,n)|_{x=1/\cos(\pi\nu)}.$$

We assume that $|n|$ is fixed. From the asymptotics of the Legendre function as $\nu \to \mathrm{i}\infty$ we have

$$|R_i(\nu,n)| \le \mathrm{const}\,|\nu|^{1/2}\left|\mathrm{e}^{\mathrm{i}\nu(\theta_1-\theta_0)}\right|,$$

$$|\mathbf{r}_i(x,n)| \le \mathrm{const}\left|\log^{1/2}(2/x)\right|\,x^{\frac{\theta_1-\theta_0}{\pi}-1},\quad x\in(0,1]. \tag{45}$$

The estimates (45) show that the Eq. 44 is to be naturally considered in $L_p(0,1)$ for $1 < p < p_* = \frac{\pi}{\pi-(\theta_1-\theta_0)}$.

Our aim in the present Appendix is to demonstrate that, *provided* $\mathfrak{Re}\,\eta > 0$ *and* n *is arbitrarily fixed, the operator in* (44) *can be represented*[5] *as* $D+K$, *where* D *is a boundedly invertible* (*Dixon*) *operator,*

$$(D\,\mathbf{r})\,(x) := \mathbf{r}(x) - \lambda\int_0^1 \frac{\mathbf{r}(y)\,\mathrm{d}y}{x+y}\,,\quad \lambda = -2\eta/\pi, \tag{46}$$

and the operator K,

$$(K\,\mathbf{r})\,(x) := \frac{\eta}{\pi}\int_0^1 \frac{\mathcal{W}(y,n)-2}{x+y}\,\mathbf{r}(y,n)\,\mathrm{d}y \tag{47}$$

is compact.

To prove the above conjecture we note that, as $\nu\to\mathrm{i}\infty$ and n is fixed, one has

$$W(\nu,n) = 2 + O_n\left(\frac{1}{\nu}\right),\quad \mathcal{W}(x,n) = 2 + O_n\left(1/\log(2/x)\right).$$

The Dixon operator D [25] is boundedly invertible ($\mathfrak{Re}\,\eta > 0$) in $L_p(0,1)$ and, moreover, the inverse can be explicitly found (see also [7]). Together with Eq. 44 for some technical simplifications we consider its equivalent (adjoint) form written for $\rho(x,n) = \mathcal{W}(x,n)\mathbf{r}(x,n)$

$$\rho(x,n) + \frac{\eta\mathcal{W}(x,n)}{\pi}\int_0^1 \frac{\rho(y,n)}{x+y}\,\mathrm{d}y = \rho_i(y,n), \tag{48}$$

$\rho_i(x,n) = \mathcal{W}(x,n)\mathbf{r}_i(x,n)$. Equations 48 and 44 have simultaneously the Fredholm property. We write (48) in the form

$$\rho(x,n) + \frac{2\eta}{\pi}\int_0^1 \frac{\rho(y,n)}{x+y}\,\mathrm{d}y + (K'\rho)(x,n) = \rho_i(x,n), \tag{49}$$

$$\left(K'\rho\right)(x) = \eta\,\frac{\mathcal{W}(x,n)-2}{\pi}\int_0^1 \frac{\rho(y)\,\mathrm{d}y}{x+y}.$$

It is demonstrated in the following section that the operator K' (and therefore K) is compact. Thus we conclude that the operator $D+K$ is Fredholm and (44) is uniquely solvable for any n.

[5] This kind of splitting was discussed in [7] in $L_2(0,1)$.

Compactness of the integral operator

We study compactness of the operator K' in (49). We exploit herein the known criterion of compactness of a set in $L_p(0,1)$. Recall that a set $T \subset L_p(0,1)$ is compact if it is bounded and for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $|t| < \delta$

$$\|g(x+t) - g(x)\,;\, L_p(0,1)\| < \epsilon$$

for all $g \in T$.

The operator K' is bounded in $L_p(0,1)$. One has

$$\begin{aligned}
&\| \left(K'\mathbf{r}\right)(x+t) - \left(K'\mathbf{r}\right)(x)\,;\, L_p(0,1)\|^p \\
&= \frac{\eta}{\pi}\int_0^1 \mathrm{d}x \left|(W(x,n) - 1)\int_0^1 \mathbf{r}(y,n)\left(\frac{1}{x+y+t} - \frac{1}{x+y}\right)\mathrm{d}y\right|^p \\
&\le c\int_0^1 \mathrm{d}x \left|\frac{1}{\log\left(\frac{x+1}{x}\right)}\right|^p \left|\int_0^1 \mathrm{d}y\left(\frac{r(y)}{x+y+t} - \frac{r(y)}{x+y}\right)\right|^p \\
&\le c_1\,\|\mathbf{r}\,;\, L_p(0,1)\|^p \int_0^1 \mathrm{d}x\,\frac{1}{\log^p\left(\frac{x+1}{x}\right)}\left[\int_0^1 \mathrm{d}y\left|\frac{1}{x+y+t} - \frac{1}{x+y}\right|^q\right]^{p/q},
\end{aligned} \tag{50}$$

where we used the Hölder inequality with $1/p + 1/q = 1$. It remains to verify that for any $\epsilon > 0$ one can take δ such that for $|t| < \delta$ the iterated integral at the right-hand side of (50) is less than ϵ. The latter follows from the inequality $\left(q = \frac{p}{p-1}\right)$

$$\begin{aligned}
\int_0^\delta \mathrm{d}x\,\frac{1}{|\log x|^p}\left[\int_0^\delta \mathrm{d}y\left|\frac{1}{x+y+t} - \frac{1}{x+y}\right|^q\right]^{p/q} &\le C\int_0^\delta \mathrm{d}x\,\frac{1}{|\log x|^p}\left[\left(\frac{1}{x}\right)^{-1+q}\right]^{p/q} \\
&= C\int_0^\delta \frac{\mathrm{d}x}{x\,|\log x|^p} \le C_1\,1/|\log\delta|^{p-1}, \quad p > 1.
\end{aligned}$$

The case $|n| \gg 1$ and convergence of the series for the spectral functions

In order to study convergence of the series in (28), (29) we should study the behaviour of the summands in the series. To this end we turn to the properties of the operator in (44) as $|n| \gg 1$. We exploit the asymptotics of the Legendre function [26]

$$P^{-|n|}_{\nu-1/2}(\cos\theta) = \frac{(\sin\theta)^{|n|}}{\Gamma(|n|+1)}\left[\frac{\left[\left(\zeta\,\cos\theta + \sqrt{1-\zeta^2\sin^2\theta}\right)/\left(1+\zeta^2\right)\right]^{\zeta}}{\cos\theta + \sqrt{1-\zeta^2\sin^2\theta}}\right]^{|n|}\left(1 + O\left(\frac{1}{|n|}\right)\right),$$

where $|n| \gg 1$, $\zeta = \nu/n$, ζ outside a close vicinity of $1/\sin\theta$. Actually we are interested in the case $|\zeta| \le q < 1$. Taking into account the asymptotics above, for $W(\nu,n)$, we have

$$|W(\nu,n)| \le \frac{|\nu|}{|n|}\,|\psi(\zeta)|\,. \tag{51}$$

Here $\psi(\zeta)$ is bounded as $\nu \in [0, \mathrm{i}|n|/A]$, $|n| \gg 1$, A is some fixed number ($A > 1$). The function $W(\nu,n)$ remains also bounded as $\nu \to \mathrm{i}\infty$, $|n| \to \infty$, $|\nu| > |n|$. For any fixed $x \in (0,1]$ in view of (51) one has

$$\lim_{|n|\to\infty} \mathcal{W}(x,n) = 0.$$

Consider the sequence of operators K_1^n defined by the equality

$$(K_1^n\rho)(x) \;=\; \frac{\eta\mathcal{W}(x,n)}{\pi}\int_0^1 \frac{\rho(y)}{x\,+\,y}\,\mathrm{d}y.$$

On exploiting the Lebesgue theorem on the limit transition in the integral, one has

$$\|K_1^n\rho\,;\,L_p(0,1)\| \;\to\; 0,\quad |n|\to\infty,$$

for any ρ such that $\|\rho;\,L_p(0,1)\|\le 1$, i.e., the sequence of the operators K_1^n strongly tends to zero.

Thus we have the estimate

$$\|(I+K_1^n)\rho;\,L_p(0,1)\| \ge \|\rho;\,L_p(0,1)\| - \|K_1^n\rho;\,L_p(0,1)\|$$

$$\ge \|\rho\| - \|K_1^n(\rho/\|\rho\|)\|\,\|\rho\| \ge \|\rho\| - \varepsilon\|\rho\| = (1-\varepsilon)\|\rho;\,L_p(0,1)\|$$

for some $\varepsilon>0$ and for all $|n|>|n_0|$ and ρ. This also means that the operators in (44) and (48) are boundedly and uniformly invertible with respect to n.

So we have demonstrated that, *provided* $|n|$ *is sufficiently large* ($|n|\gg 1$)*, the operator in the Eq.* 48 *is boundedly invertible* (*the norm of the inverse is bounded uniformly with respect to* n) *in* $L_p(0,1)$, $1<p<p_*$.

We can conclude that for the solution of (44) as $|n|\gg 1$ the estimate is valid

$$\|\mathbf{r}\,;\;L_p(0,1)\| \;\le\; C\,\|\mathbf{r}_i\,;\;L_p(0,1)\|, \tag{52}$$

where the constant C is independent of $|n|$, θ_1, θ_0. This estimate is crucial for the proof of convergence of the series (28), (29) in $L_p(0,1)$ uniformly with respect to θ,φ,θ_0. This convergence (the details can be found in [10]) is a simple consequence of the estimate (52) and of the fact that the series for u^i_ν converge. It can be shown that the spectral functions $u_\nu,\ v_\nu$ are found to belong to the desired class of functions: they are meromorphic, are regular as $\nu\in\Pi_\delta$ and vanish exponentially as $\nu\to\mathrm{i}\infty$. Their singularities belong to the real axis on ν-plane.

Appendix B: Functional difference equation and Carleman boundary-value problem for analytic vectors

In the present Appendix we demonstrate that the problem for the functional difference equation (37) is reducible to a boundary-value problem of Carleman type [27] for analytic vectors. Indeed, we reduce the second-order functional difference equation to a system of two equations of the first order by adopting the notations

$$\psi_1(\nu)=R_u(\nu,n),\quad \psi_2(\nu)=R_u(\nu-1,n).$$

Then we find from (37)

$$\Psi(\nu+1)=A(\nu)\Psi(\nu)+\xi(\nu), \tag{53}$$

where $\Psi(\nu)=(\psi_1(\nu),\psi_2(\nu))^T$,

$$A(\nu)=\begin{pmatrix}-2\mathrm{i}\eta W(\nu,n) & 1\\ 1 & 0\end{pmatrix},\quad \xi(\nu)=\begin{pmatrix}S(\nu,n)\\ 0\end{pmatrix}.$$

A system of equations similar to (53) was studied in [17] by Buslaev and Fedotov from another point of view. The vector Ψ is regular in the strip $\Pi_1=\{\nu:\ 0<\mathfrak{Re}\,\nu<1\}$ and vanishes exponentially as $\nu\to\pm\mathrm{i}\infty$.

We apply the conformal mapping of the strip Π_1 onto the upper half-plane $\tau=\exp(\mathrm{i}\pi\nu)$ so that the left boundary of the strip is mapped onto $\mathbb{R}_+$ and we arrive at the Carleman-type boundary-value problem for analytic vectors

$$\psi(\alpha(\tau))=a(\tau)\psi(\tau)+\xi_0(\tau),\quad \tau\in\mathbb{R}_+ \tag{54}$$

where $\alpha(\tau)=-\tau,\ \psi(\tau)=\Psi(\nu(\tau))$ and $a(\tau)=A(\nu(\tau))$, $\xi_0(\tau)=\xi(\nu(\tau))$, $\alpha(\alpha(\tau))=\tau$. The problem is to find an analytic vector in the upper half-plane satisfying the boundary condition (54) and vanishing as $\tau\to 0$ and $\tau\to\infty$.

The Carleman problem is known to be reducible to integral equations [27]; however, we do not intend to exploit this reduction.

References

1. Felsen LB (1957) Plane wave scattering by small-angle cones. IRE Trans Antennas Propagation 5:121–129
2. Borovikov VA (1966) Diffraction by polygons and polyhedrons. Nauka, Moscow
3. Cheeger J, Taylor ME (1982) Diffraction of waves by conical singularities. Commun Pure Appl Math 35(3,4): 275–331, 487–529
4. Smyshlyaev VP (1990) Diffraction by conical surfaces at high frequencies. Wave Motion 12(4):329–339
5. Jones DS (1997) Scattering by a cone. Quart J Mech Appl Math 50:499–523
6. Babich VM, Dement'ev DB, Samokish BA, Smyshlyaev VP (2000) On evaluation of the diffraction coefficients for arbitrary 'non-singular' directions of a smooth convex cone. SIAM Appl Math 66(2):536–573
7. Bernard J-ML (1997) Méthode analytique et transformées fonctionnelles pour la diffraction d'ondes par une singularité conique: équation intégrale de noyau non oscillant pour le cas d'impédance constante. rapport CEA-R-5764, Editions Dist-Saclay
8. Bernard J-ML, Lyalinov MA (2001) Diffraction of acoustic waves by an impedance cone of an arbitrary cross-section. Wave Motion 33:155–181 (erratum: p 177 replace $O(1/\cos(\pi(\nu-b)))$ by $O(\nu^d \sin(\pi\nu)/\cos(\pi(\nu-b))))$
9. Antipov YA (2002) Diffraction of a plane wave by a circular cone with an impedance boundary condition. SIAM J Appl Math 62(4):1122–1152
10. Lyalinov MA (2003) Diffraction of a plane wave by an impedance cone. Zap Nauch Sem Peterburg Otd Matem Inst Steklova) 297 Matem vopr teorii rasprostr voln 32:191–215 [English transl in: J Math Sci 2005, 127(6):2446–2460]
11. Bernard J-ML, Lyalinov MA (2004) Electromagnetic scattering by a smooth convex impedance cone. IMA J Appl Math 69:285–333
12. Daniele VG (2003) Winer–Hopf technique for impenetrable wedges having arbitrary aperture angle. SIAM J Appl Math 63(4):1442–1460
13. Abrahams ID, Lawrie JB (1995) On the factorization of a class of Winer–Hopf kernels. IMA J Appl Math 55(1):35–47
14. Antipov YA, Silvestrov VV (2004) Second-order functional-difference equations I: method of the Riemann–Hilbert problem on Riemann surfaces. Quart J Mech Appl Math 57(2):245–265
15. Lyalinov MA, Zhu NY (2003) A solution procedure for second-order difference equations and its application to electromagnetic-wave diffraction in a wedge-shaped region. Proc Roy Soc Lond A 459:3159–3180
16. Lyalinov MA, Zhu NY (2006) Diffraction of a skew incident plane electromagnetic wave by an impedance wedge. Wave Motion 44:21–43
17. Buslaev VS, Fedotov AA (2001) On the difference equations with periodic coefficients. Adv Theor Math Phys 5:1105–1168
18. Senior TBA, Volakis JL (1995) Approximate boundary conditions in electromagnetics. Inst of Electr Eng, London
19. Buldyrev VS, Lyalinov MA (2001) Mathematical methods in modern electromagnetic diffraction theory, vol 1. Intern monographs on advanced electromagnetics Science House, Tokyo
20. Abramowitz M, Stegun I (1972) Handbook of mathematical functions. Dover Publications
21. Gradshteyn IS, Ryzhik IM (1980) Table of integrals, series and products, 4th edn. Academic Press, Orlando
22. Jones DS (1964) The theory of electromagnetism. Pergamon Press, London
23. Tuzhilin AA (1973) The theory of Maliuzhinets' inhomogeneous functional equations. Different Uravn 9(10):1875–1888
24. Jones DS (1980) The Kontorovich–Lebedev transform. J Inst Math Appl 26:133–141
25. Titchmarsh EC (1937) Introduction to the theory of Fourier integrals. Oxford Press
26. Babich VM (2002) To the problem on the asymptotics with respect to indices of the associated Legendre functions. Russian J Math Phys 9(1):6–13 (in English)
27. Vekua NP (1970) Systems of singular integral equations. Nauka, Moscow (1st edn. 1967 Noordhoff Groningen)